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Symmetries, first integrals, and the inverse problem of Lagrangian mechanics: II

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Abstract. This paper investigates under what conditions the knowledge of a symmetry and a first integral of a linear system of second-order differential equations provides information about the existence of a Lagrangian. The relationship between symmetries and first integrals through Noether's theorem thereby plays an essential role. The treatment being confined to linear systems and linear symmetries, it is shown that there is no loss of generality in taking only quadratic Lagrangians into consideration.

1. Introduction

In a number of recent contributions we have investigated various aspects of the search for symmetries and first integrals on the one hand (Sarlet and Cantrijn 1981a, b, Cantrijn and Sarlet 1981) and the so-called inverse problem of Lagragian mechanics on the other hand (Sarlet *et al* 1982, Sarlet 1982). As a natural outcome of these studies we have further been led to look at various possible connections between both problems (Sarlet 1981, 1983a). The present paper is a continuation of a previous one with the same title (Sarlet 1981), in the following referred to as I. Needless to say, we have not been the only contributors to these areas. We limit ourselves here to citing a few closely related papers and hope that the reader will further sift his way through the literature with the help of the references in these papers. Concerning related studies on symmetries and first integrals we can mention for example: Crampin (1977), Djukic (1974), Gonzales-Gascon and Rodriguez-Camino (1980a, b, c), Leach (1981), Prince (1981), Steeb *et al* (1981). For the inverse problem of Lagrangian mechanics see e.g. Crampin (1981), de Ritis *et al* (1980), Henneaux (1982a, b), Marmo and Saletan (1977). Finally, concerning the problem of linking both issues, see also Schafir (1981, 1982), Takens (1977).

Our own view of the 'linking question', as initiated in I, has been largely dominated by Noether's theorem which in a proper setting, rather than establishing a direct, rectilinear relation between symmetry generators Y and first integrals F, reveals a triangular structure between Y, F and the Lagrangian L. Indeed, the concept of a Noether symmetry only makes sense with respect to a given Lagrangian L for the problem, in the sense that it is an invariance transformation of the two-form $d\theta(L)$, $\theta(L)$ being the Cartan form corresponding to L. The symmetry–first-integral duality

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is then completely determined by the formula

$$i_Y \, \mathrm{d}\theta(L) = \mathrm{d}F, \tag{1}$$

where i_Y denotes contraction of forms with the vector field Y. Now, changing the Lagrangian (if possible) alters the Noether symmetries corresponding to given first integrals. As a result, one can raise the new question whether knowledge of a symmetry Y of the equations of motion and a suitably related first integral F may be a deterministic element in finding a Lagrangian governing the given differential equations. This question was completely solved for the case of one degree of freedom in I. For reasons of brevity we refer to §§ 1 and 2 of I for more details about the above considerations and for an extensive description of the notations and terminology which will also be adopted here. We merely repeat here that for second-order equations

$$\ddot{q}^i = \Lambda^i(t, q, \dot{q}), \tag{2}$$

which are governed by the vector field

$$\Gamma = \partial/\partial t + \dot{q}^i \, \partial/\partial q^i + \Lambda^i \, \partial/\partial \dot{q}^i, \tag{3}$$

we write symmetry generators Y in the form

$$Y = \mu^i \, \partial/\partial q^i + \nu^i \, \partial/\partial \dot{q}^i. \tag{4}$$

The symmetry condition $[Y, \Gamma] = 0$ then requires that

$$\nu^i = \Gamma(\mu^i), \qquad \Gamma(\nu^i) = Y(\Lambda^i). \tag{5}$$

For multiple degree-of-freedom systems, it is certainly out of the question to count on results which are as strong as in I, where every pair (Y, F) satisfying the necessary condition $Y(F) = 0$ was shown to be Noether interrelated with respect to some Lagrangian. Therefore, one may in the first place wonder whether for a given *fixed* symmetry Y one can identify conditions on a related first integral F which would make the pair (Y, F) Noether interrelated with respect to some L. The first propositions in § 3 provide a result of that nature.

Immediately, however, we know that such conditions on F, assuming they would always be identifiable, will often have no solutions. The two-dimensional Kepler problem provides a simple example in that respect. Indeed, the non-Noether symmetry discussed by Prince and Eliezer (1981) can never become a Noether symmetry with respect to some other Lagrangian, because the Lagrangian for that problem is essentially unique (see Henneaux 1982b, Sarlet 1982). For that reason we must somewhat widen the scene by varying Y. So, the next question to ask is: under what conditions for Y (or F) does there exist a suitably related F (respectively Y) leading to a corresponding Lagrangian L?

In this paper we analyse these questions for the special case of linear systems. Most of the statements in the main sections have already been announced without proof in Sarlet (1983b). Section 2 contains a preliminary review of results from previous studies on linear systems, concerning both the search for symmetries and first integrals and the search for multipliers in the inverse problem of Lagrangian mechanics.

Before proceeding, we wish to recall another previous result which provides a useful general criterion for dynamical symmetries to become Noether symmetries with respect to some Lagrangian L. It will repeatedly be used in proving later theorems.

Lemma 1. If the second-order system (1) is derivable from a Lagrangian L, then a dynamical symmetry Y (of the form (4)) will be of Noether type with respect to L if and only if

$$(\partial^2 L/\partial \dot{q}^i\, \partial \dot{q}^j)\mu^j = -\partial F/\partial \dot{q}^i \tag{6}$$

for some first integral $F(t, q, \dot{q})$.

This lemma was also stated in I in a slightly different version. The proof of it can be found in Sarlet and Cantrijn (1981a).

2. Preliminaries on linear systems

We now restrict the functions Λ^i in (2) and (3) to be linear in q and $\dot{q}$. Explicitly, we consider systems of the form

$$\ddot{q} + 2A(t)\dot{q} + B(t)q = 0, \tag{7}$$

where A and B are arbitrary time-dependent $n\times n$ matrices (sufficiently smooth). Because of the linearity of (7) we further restrict our investigations to linear symmetry generators, that is, for a Y of the form (4) we assume that the components μ^i and ν^i are linear expressions in q and $\dot{q}$, say

$$\mu = -(P(t)q + Q(t)\dot{q}) \tag{8}$$

for some matrices P and Q. The symmetry requirements (5) in that case mean that P and Q should satisfy the following set of coupled second-order matrix equations:

$$\ddot{Q} = 4(\dot{Q}A + AQA - QA^2) + 2(Q\dot{A} - A\dot{Q}) + QB - BQ + 2(PA - AP - \dot{P}), \tag{9}$$

$$\ddot{P} = 2(\dot{Q}B + AQB - QAB) + Q\dot{B} - 2A\dot{P} + PB - BP. \tag{10}$$

Note that selecting a particular solution of (9) and (10) and defining accordingly a vector μ by (8) completely determines a symmetry Y, the remaining components ν^i being determined by the first of equations (5).

An interesting special case is the case of point symmetries (that is, the $\partial/\partial t$ component τ and the $\partial/\partial q^i$ components ξ^i, in the notation of I, do not depend upon the velocities). For linear symmetries, being of point type essentially means either that Q is the unit matrix (to within a rescaling of the time variable), or that Q is zero. In the first case, the conditions (9) and (10) can be shown to reduce to

$$\dot{P} - \dot{A} = [P, A], \tag{11a}$$

$$\Phi^{(1)} = [\Phi^{(0)}, P - A], \tag{12a}$$

whereas in the case $Q = 0$ we get

$$\dot{P} = [P, A], \tag{11b}$$

$$[\Phi^{(0)}, P] = 0. \tag{12b}$$

The matrices $\Phi^{(0)}(t)$ and $\Phi^{(1)}(t)$ are defined by

$$\Phi^{(0)} = B - A^2 - \dot{A}, \tag{13}$$

$$\Phi^{(1)} = \dot{\Phi}^{(0)} + [A, \Phi^{(0)}], \tag{14}$$

and the bracket denotes the usual commutator of matrices.

Since we want to find out how dynamical symmetries of type (8) can become Noether symmetries with respect to some Lagrangian, let us now mention some basic results about the inverse problem of Lagrangian mechanics. For linear systems like (7) it is natural to think in the first place of quadratic Lagrangians. The problem then consists in finding a non-singular symmetric multiplier matrix $V(t)$ which gives (7) the structure of Euler–Lagrange equations. Such a V must satisfy (see Sarlet *et al* 1982)

$$\dot{V} = A^{\mathrm{T}} V + VA, \tag{15}$$

$$V\Phi^{(0)} = (V\Phi^{(0)})^{\mathrm{T}}. \tag{16}$$

One may conceive other than quadratic Lagrangians for (7), but it is worthwhile knowing that, if a non-quadratic Lagrangian for (7) exists, there certainly exists a quadratic one too (Sarlet 1983c). We will come back to this in greater detail in § 4.

If a linear symmetry is of Noether type with respect to some quadratic Lagrangian, the corresponding Noether invariant obviously will be quadratic too. So let us finally describe a set of equations which govern the search for quadratic first integrals of (7). We write a general quadratic F in the form

$$F = \tfrac{1}{2}(\dot{q}^{\mathrm{T}} V_1(t)\dot{q} + q^{\mathrm{T}} D_1(t)\dot{q} + q^{\mathrm{T}} Z_1(t) q), \tag{17}$$

where V_1 and Z_1 are symmetric. Imposing the condition that F be a first integral ($\Gamma(F) = 0$) would of course lead to a set of matrix differential equations for V_1, D_1 and Z_1. It will be useful, however, to make use of an equivalent set of equations in which the coefficients exhibit the same quantities $\Phi^{(0)}$ and $\Phi^{(1)}$ encountered above. The equations were derived in Sarlet and Bahar (1981) and read

$$\mathscr{D}_A V_1 = W_1, \qquad \mathscr{D}_A W_1 = U_1, \tag{18a, b}$$

$$\mathscr{D}_A U_1 = (R_1 - 2W_1)\Phi^{(0)} - V_1\Phi^{(1)} + \text{transpose}, \tag{18c}$$

$$\mathscr{D}_A R_1 = (V_1\Phi^{(0)})^{\mathrm{T}} - V_1\Phi^{(0)}, \tag{18d}$$

where the operator $\mathscr{D}_A$ is defined by

$$\mathscr{D}_A X = \dot{X} - (A^{\mathrm{T}} X + XA) \tag{19}$$

and we have $V_1 = V_1^{\mathrm{T}}$, $W_1 = W_1^{\mathrm{T}}$, $U_1 = U_1^{\mathrm{T}}$, $R_1 = -R_1^{\mathrm{T}}$. In terms of these matrices, D_1 and Z_1 are then determined by

$$D_1 = 2A^{\mathrm{T}} V_1 - W_1 + R_1, \tag{20}$$

$$2Z_1 = V_1\Phi^{(0)} + \tfrac{1}{2}U_1 + A^{\mathrm{T}} V_1 A - (W_1 - R_1)A + \text{transpose}. \tag{21}$$

3. Relating linear symmetries to quadratic Lagrangians

We now return to an investigation of the two types of questions which were announced in the introduction. Basically we wish to look for situations in which linear dynamical symmetries Y of (7) turn out to be Noether symmetries with respect to some Lagrangian. For the time being, we thereby only take quadratic Lagrangians into consideration. It will be shown in § 4 that this does not create a loss of generality.

For the first result we consider a *fixed*, given *point symmetry* Y and look for conditions on a suitably related F which will guarantee that the pair (Y, F) is Noether interrelated with respect to some quadratic L.

Proposition 1a. Let $\mu = -(Pq + \dot{q})$ determine a *dynamical symmetry* Y for (7) and put

$$F = \tfrac{1}{2}(\dot{q}^{\mathrm{T}} V \dot{q} + 2q^{\mathrm{T}} P^{\mathrm{T}} V \dot{q} + q^{\mathrm{T}} Z q), \tag{22}$$

V and Z being symmetric matrices (with V non-singular). Then (7) has a quadratic Lagrangian L (multiplier V) and Y is a Noether symmetry with respect to L, if and only if

$$\Gamma(F) = 0 \qquad \text{and} \qquad Y(F) = 0.$$

Proof. The 'only if' part is trivial. Indeed, any quadratic L for (7) is of the form (see Sarlet *et al* 1982)

$$L = \tfrac{1}{2}[\dot{q}^{\mathrm{T}} V \dot{q} + 2q^{\mathrm{T}} A^{\mathrm{T}} V \dot{q} + \tfrac{1}{2} q^{\mathrm{T}}(A^{\mathrm{T}} V A - V\Phi^{(0)})q], \tag{23}$$

with V satisfying (15) and (16). So, if Y is of Noether type with respect to such an L, the corresponding Noether invariant following from (1) is of the form (22) and will also satisfy the condition $Y(F) = 0$.

For the 'if' part, assume now that Y is a dynamical symmetry and F satisfies both additional requirements. Expressing $Y(F) = 0$ for a quadratic F will in general lead to three matrix conditions. For an F of type (22), however, one can easily show that one of these conditions is identically satisfied in view of the other two and the latter can be written as

$$V(A - P) + (A^{\mathrm{T}} - P^{\mathrm{T}})V = 0, \tag{24}$$

$$Z = 2P^{\mathrm{T}} V A - P^{\mathrm{T}} V P + B^{\mathrm{T}} V - \dot{P}^{\mathrm{T}} V - (P^{\mathrm{T}})^2 V. \tag{25}$$

Equation (24) implies that

$$P - A = -\tfrac{1}{2} V^{-1} R \qquad \text{with } R^{\mathrm{T}} = -R. \tag{26}$$

We further know from our assumptions that P satisfies (11a) and (12a). Using (11a) to replace $\dot{P}$ in (25), this in the first place allows us to rewrite Z as

$$Z = A^{\mathrm{T}} V A + \tfrac{1}{2} R A - \tfrac{1}{2} A^{\mathrm{T}} R + \Phi^{(0)\mathrm{T}} V. \tag{27}$$

Secondly, (11a) and (12a), in view of (26), imply that

$$\tfrac{1}{2} V^{-1} \dot{V} V^{-1} R - \tfrac{1}{2} V^{-1} \dot{R} = \tfrac{1}{2}[A, V^{-1} R], \tag{28}$$

$$\Phi^{(1)} = \tfrac{1}{2}[V^{-1} R, \Phi^{(0)}]. \tag{29}$$

Next, we try to express the condition $\Gamma(F) = 0$. To that end, the present expression (22) for our F, with Z given by (27), should match the general expression (17) for a quadratic first integral, supplemented by conditions (18), (20) and (21). Looking at the general form (20) for D_1 and taking account of the present condition (26), it is clear that V and R correspond to V_1 and R_1 respectively, while the matrix W_1 in our case must be zero.

Equation (18b) then further implies $U_1 = 0$ so that, all together, $\Gamma(F) = 0$ requires that V and R satisfy the equations

$$\mathscr{D}_A V = 0, \tag{30}$$

$$0 = R\Phi^{(0)} - V\Phi^{(1)} + \text{transpose}, \tag{31}$$

$$\mathscr{D}_A R = (V\Phi^{(0)})^T - V\Phi^{(0)}, \tag{32}$$

while Z, in view of (21), must be of the form

$$Z = A^T VA + \tfrac{1}{2}RA - \tfrac{1}{2}A^T R + \tfrac{1}{2}V\Phi^{(0)} + \tfrac{1}{2}\Phi^{(0)T} V. \tag{33}$$

Now, replacing $\dot{V}$ in (28) from the relation (30), one finds $\mathscr{D}_A R = 0$, so that (32) further implies that $V\Phi^{(0)}$ is symmetric. This means that the conditions (15) and (16) for a multiplier are met. Moreover, one can further check that (29) becomes consistent with (31) and the two expressions (27) and (33) for Z are eventually identical. V now being a multiplier, it is finally straightforward to verify that the corresponding Lagrangian L (see (23)), the symmetry components μ^i and the first integral F are related by a formula of type (6), which according to lemma 1 implies that Y ultimately is a Noether symmetry.

A similar result can be stated for the class of point symmetries in which $Q = 0$ (see (11*b*), (12*b*)).

Proposition 1b. Let $\mu = -Pq$, with P non-singular, determine a dynamical symmetry Y for (7) and put

$$F = q^T P^T V\dot{q} + \tfrac{1}{2}q^T Zq,$$

V and Z being symmetric (with V non-singular). Then (7) has a quadratic Lagrangian L (multiplier V) and Y is a Noether symmetry with respect to L, if and only if

$$\Gamma(F) = 0 \qquad \text{and} \qquad Y(F) = 0.$$

Let us now move on to the second type of question, mentioned in the introduction, where we no longer start with a fixed Y or F, but look for conditions on Y or F which will ensure the existence of a corresponding F or Y in the Noether picture.

The following result should be regarded as providing sufficient conditions for the structure of suitable quadratic first integrals.

Proposition 2. Let $V(t)$ be a non-singular symmetric matrix and $R(t)$ a skew-symmetric matrix satisfying $\mathscr{D}_A R = 0$. Define a matrix P by $P = A - \tfrac{1}{2}V^{-1}R$ and a quadratic function F by

$$F = \tfrac{1}{2}(\dot{q}^T V\dot{q} + 2q^T P^T V\dot{q} + q^T Zq),$$

with Z defined by (33). Then V is a multiplier for (7) and $\mu = -(Pq + \dot{q})$ generates a dynamical symmetry if and only if $\Gamma(F) = 0$. The symmetry generated by μ moreover is a Noether symmetry with respect to the Lagrangian corresponding to V.

Remark. It is clear that the ingredients in this statement are somehow the same as those in the statement and proof of proposition 1a. Some of the intermediate implications encountered in the previous proof are now taken as assumptions, and as such essentially replace the previous assumption $Y(F) = 0$. We therefore leave the explicit proof of proposition 2 as an exercise for the reader.

Let us now turn to general linear symmetries (not necessarily of point type) and establish a result which this time characterises suitable symmetries Y for which a related F exists.

Proposition 3. In order that (7) have a multiplier, it is sufficient that there exist a linear symmetry determined by (8), with Q and P of the form

$$Q = V^{-1}V_1, \tag{34}$$

$$P = QA - \tfrac{1}{2}V^{-1}(W_1 + R_1), \tag{35}$$

where V_1, W_1 and R_1 satisfy equations (18), V is an as yet arbitrary non-singular symmetric matrix, and the following two regularity conditions hold true:

$$\det V_1 \neq 0, \qquad \det(Z_1 - \tfrac{1}{4}D_1 V_1^{-1} D_1^{\mathrm{T}}) \neq 0, \tag{36a, b}$$

D_1 and Z_1 being defined by (20) and (21).

Proof. We set

$$\mathrm{d}(V^{-1})/\mathrm{d}t = -V^{-1}A^{\mathrm{T}} - AV^{-1} + W, \tag{37}$$

which defines a new matrix W. From the assumptions, we know that Q and P, as defined by (34), (35), satisfy the symmetry conditions (9) and (10). Substituting Q and P into (9) and making use of (18) and (37) produces the following relation:

$$(\dot{W} + AW + WA^{\mathrm{T}})V_1 = W(R_1 - W_1) + (V^{-1}\Phi^{(0)\mathrm{T}} - \Phi^{(0)}V^{-1})V_1. \tag{38}$$

The analogous computation for (10) is rather tedious, but straightforward. If one further makes use of (38), the end result simplifies to

$$\tfrac{1}{2}[\dot{W} + AW + WA^{\mathrm{T}} + \Phi^{(0)}V^{-1} - \Phi^{(0)\mathrm{T}}V^{-1}](W_1 + R_1) + W(U_1 + V_1\Phi^{(0)} + \Phi^{(0)\mathrm{T}}V_1) = 0. \tag{39}$$

Now, from the assumption that V_1 is non-singular, one can solve (38) for the quantity in square brackets in (39), by which (39) further reduces to

$$W[\tfrac{1}{2}(W_1 - R_1)V_1^{-1}(W_1 + R_1) - U_1 - V_1\Phi^{(0)} - \Phi^{(0)\mathrm{T}}V_1] = 0. \tag{40}$$

One readily checks that the matrix multiplying W in (40) is $-2Z_1 + \tfrac{1}{2}D_1V_1^{-1}D_1^{\mathrm{T}}$, so that (40) and (36$b$) imply $W = 0$. Equations (37) and (38) then show that the symmetric matrix V satisfies the conditions (15) and (16), and therefore is a multiplier in the inverse problem.

Remarks

(i) Equations (18) always have solutions for V_1, W_1, U_1, R_1, and as such produce a quadratic first integral F for (7). So the restrictions embodied in the statement of proposition 3 come from expressing that a Q and a P of the form (34), (35), satisfy the symmetry requirements (9), (10). Note also that an application of lemma 1 again shows that the dynamical symmetry we started with will eventually be of Noether type with respect to the Lagrangian corresponding to the multiplier V.

(ii) The regularity assumptions (36) have a nice interpretation. Indeed, the quadratic first integral F can be written in the form $F = \tfrac{1}{2}x^{\mathrm{T}}\Delta x$, with $x = \mathrm{col}(q, \dot{q})$ and

$$\Delta = \begin{pmatrix} Z_1 & \tfrac{1}{2}D_1 \\ \tfrac{1}{2}D_1^{\mathrm{T}} & V_1 \end{pmatrix}.$$

If V_1 is non-singular, one can show that

$$\det \Delta = (\det V_1) \det(Z_1 - \tfrac{1}{4} D_1 V_1^{-1} D_1^{\mathrm{T}}),$$

so that (36) means that F is a non-degenerate quadratic form in x space.

4. What about non-quadratic Lagrangians?

As mentioned before, it has been shown (Sarlet 1983c) that every linear system having a non-quadratic Lagrangian has a quadratic one too. Yet this property does not in principle exclude the possibility that a linear symmetry Y, which would be of non-Noether type with respect to all quadratic Lagrangians, might be a Noether symmetry with respect to a non-quadratic Lagrangian for (7) (assuming such a Lagrangian exists). So it would certainly add more weight to the results of § 3 if we could show that the restriction to quadratic Lagrangians there does not really involve a loss of generality. We will do that now by proving the following theorem.

Proposition 4. Assume that a μ of the form (8) determines a *dynamical symmetry* Y of (7). If Y is a Noether symmetry with respect to some (possibly non-quadratic) Lagrangian L, then there exists a related linear symmetry Y' of (7) and a related quadratic Lagrangian L' such that Y' is a Noether symmetry with respect to L'.

In order to prove this statement, it is easier to pass to the canonical form of (7). We therefore introduce the matrix $\Omega(t)$, which is the solution of the following problem:

$$\dot{\Omega} + A\Omega = 0, \qquad \Omega(t_0) = \mathbb{1}, \tag{41}$$

$\mathbb{1}$ being the $n \times n$ unit matrix. The transformation $q = \Omega q'$ reduces (7) to

$$\ddot{q}' + \overline{\Phi^{(0)}} q' = 0, \tag{42}$$

where the bar is used for the similarity transformation

$$\bar{X} = \Omega^{-1} X \Omega. \tag{43}$$

It is known that point transformations preserve the Lagrangian character of second-order equations. Moreover, symmetries and first integrals for one equation will be carried over in a natural way to symmetries and first integrals for the transformed equation. The following lemma will therefore be intuitively clear. Nevertheless, it is of some interest to carry out a complete proof for the sake of having explicit formulae linking results for both systems.

Lemma 2. Proposition 4 holds true for (7) if and only if it is true for the reduced system (42).

Proof. (i) For any μ, defined by (8), we introduce

$$\mu' = -(P'q' + Q'\dot{q}'), \tag{44}$$

with $Q' = \bar{Q}$, $P' = \bar{P} - \bar{Q}\bar{A}$. It is then straightforward to check that P and Q satisfy (9) and (10), if and only if P' and Q' satisfy the corresponding symmetry requirements for the reduced system (42).

(ii) If L is a Lagrangian for (7), corresponding to a multiplier matrix $\alpha(t, q, \dot{q})$, then a Lagrangian for (42) is determined by the multiplier

$$S(t, q', \dot{q}') = \Omega^{\mathrm{T}} \alpha \Omega, \tag{45}$$

where α of course is expressed in terms of the new variables.

(iii) $F(t, q, \dot{q})$ is a first integral for (7) if and only if

$$F'(t, q', \dot{q}') = F(t, \Omega q', -A\Omega q' + \Omega \dot{q}') \tag{46}$$

is a first integral for (42).

(iv) Assume we have

$$\alpha\mu = -\partial F/\partial \dot{q};$$

then, upon multiplication by Ω^{T} and expressing everything in terms of the new variables, we get

$$S\mu' = -\partial F'/\partial \dot{q}'.$$

Lemma 2 now follows from the observations (i)–(iv) and an application in both directions of the fundamental lemma 1.

Now that we know in detail how to convert results for (7) to corresponding results for (42) and *vice versa*, we now concentrate on equations of type (42) and, for simplicity in notation, we omit all bars and dashes. Alternatively speaking, we consider equations of the form (7) in the special case that $A = 0$ and B is denoted by $\Phi^{(0)}$. Recall that now we are no longer exclusively interested in quadratic Lagrangians, in other words, conditions (15) and (16) no longer suffice. Concerning the theory for more general multipliers, see e.g. Sarlet (1982). We here confine ourselves to citing the relevant equations which will be satisfied by any multiplier of (42).

Lemma 3. A non-singular symmetric matrix $S(t, q, \dot{q})$ is a multiplier in the inverse problem related to (42), iff S satisfies the following equations:

$$\partial S_{ij}/\partial \dot{q}^k = \partial S_{ik}/\partial \dot{q}^j, \qquad \partial S_{ij}/\partial q^k = \partial S_{ik}/\partial q^j, \qquad \forall i, j, k, \tag{47a, b}$$

$$S(t, q, \dot{q})\Phi^{(0)}(t) = [S(t, q, \dot{q})\Phi^{(0)}(t)]^{\mathrm{T}}, \tag{48}$$

$$\Gamma(S) = 0, \tag{49}$$

where Γ, as usual, denotes the vector field governing the system (42).

Remarks

(i) Equation (47b) is actually a direct consequence of (47a) and (49), as can easily be verified. With reference to the general treatment in Sarlet (1982), it is in fact one of the closure conditions which is always identically satisfied.

(ii) One can show that any multiplier for (42) must be of the form

$$S(t, q, \dot{q}) = \sum_{i=1}^{k} f_i(t, q, \dot{q}) S_i,$$

where the k matrices S_i are linearly independent constant multipliers (and as such yield quadratic Lagrangians) and the functions f_i are constants of the motion. We are of course only interested here in those cases where multipliers S_i exist.

Lemma 4. Let $\mu = -(P(t)q + Q(t)\dot{q})$ define a linear dynamical symmetry Y for (42). If Y is a Noether symmetry with respect to some Lagrangian L (with corresponding multiplier S), then we have the following properties:

$$\text{(i)} \qquad S(t, q, \dot{q})Q(t) \text{ is symmetric,} \tag{50}$$

$$\text{(ii)} \qquad S(t, q, \dot{q})(\dot{P} - Q\Phi^{(0)})(t) \text{ is symmetric,} \tag{51}$$

$$\text{(iii)} \qquad S\dot{Q} + SP + P^{\mathrm{T}}S = Y(S). \tag{52}$$

Proof. If Y is a Noether symmetry, then we have

$$-S\mu = \partial F/\partial \dot{q} \tag{53}$$

for some first integral F. Computing Γ of both sides of (53) and taking account of $\Gamma(F) = 0$ and $\Gamma(S) = 0$, we get

$$S\nu = \partial F/\partial q, \tag{54}$$

with

$$\nu = \Gamma(\mu) = -(\dot{Q} + P)\dot{q} + (Q\Phi^{(0)} - \dot{P})q. \tag{55}$$

The integrability conditions $\partial^2 F/\partial \dot{q}^i\, \partial \dot{q}^j = \partial^2 F/\partial \dot{q}^j\, \partial \dot{q}^i$ through (53) yield

$$(\partial S_{ik}/\partial \dot{q}^j)(P_{kl}q^l + Q_{kl}\dot{q}^l) + S_{ik}Q_{kj} = (\partial S_{jk}/\partial \dot{q}^i)(P_{kl}q^l + Q_{kl}\dot{q}^l) + S_{jk}Q_{ki}.$$

The property (47*a*) of S then immediately leads to (50). The other integrability conditions for (53) and (54) using (47*a*, *b*) similarly lead to (51) and (52).

Now, if we arbitrarily fix the values of t, q and $\dot{q}$ in S, we get a constant solution of the inverse problem equations mentioned in lemma 3. This is not immediately apparent concerning the algebraic condition (48). It can be shown however, that (48), under the assumption that all given matrices are real analytic, can be replaced by the infinite set of conditions

$$S(t, q, \dot{q})\Phi^{(k)}(t_0) = [S(t, q, \dot{q})\Phi^{(k)}(t_0)]^{\mathrm{T}}, \qquad k = 0, 1, \ldots, \infty, \tag{56}$$

where the matrices $\Phi^{(k)}(t)$ are recursively defined as in (14) and in the present situation are in fact simply the time derivatives of $\Phi^{(0)}(t)$. Looking at (56) instead of (48), it is indeed clear that

$$S_0 = S(t_0, q_0, \dot{q}_0) \tag{57}$$

will be a constant multiplier of (42) and as such will give rise to a quadratic Lagrangian L_0. From lemma 4 it trivially follows that $S_0Q(t_0)$ and $S_0(\dot{P} - Q\Phi^{(0)})(t_0)$ are symmetric and we have

$$S_0\dot{Q}(t_0) + S_0P(t_0) + P^{\mathrm{T}}(t_0)S_0 = \Sigma_0,$$

with

$$\Sigma_0 = Y(S)(t_0, q_0, \dot{q}_0). \tag{58}$$

We next want to show that these properties remain true when S_0 and Σ_0 are kept fixed, whilst the matrices Q, P and $\Phi^{(0)}$ are evaluated at any time t.

Lemma 5. If Y is a Noether symmetry with respect to L (multiplier $S(t, q, \dot{q})$), we have for all t:

(i) $S_0Q(t)$ is symmetric, (59)

(ii) $S_0(\dot{P} - Q\Phi^{(0)})(t)$ is symmetric, (60)

(iii) $S_0\dot{Q}(t) + S_0P(t) + P^{\mathrm{T}}(t)S_0 = \Sigma_0$. (61)

Proof. As a preliminary step we notice that S_0, being itself a multiplier, satisfies the condition (48), that is

$$S_0\Phi^{(0)}(t) = (S_0\Phi^{(0)}(t))^{\mathrm{T}}. \tag{62}$$

Moreover, letting the vector field Y act on (56), fixing subsequently the values of t, q and $\dot{q}$ and recombining all resulting relations to get the Taylor expansion of $\Phi^{(0)}(t)$, we also conclude that

$$\Sigma_0\Phi^{(0)}(t) = (\Sigma_0\Phi^{(0)}(t))^{\mathrm{T}}, \tag{63}$$

for all t. Now, as noted above, we know the validity of (59)–(61) for $t = t_0$. Equation (61) at $t = t_0$ further implies that $S_0\dot{Q}_0(t)$ is symmetric. Next, from the symmetry requirement (9) (with $A = 0$ and $B = \Phi^{(0)}$) and making use of (62), we can write

$$\begin{aligned} S_0\ddot{Q} &= S_0(Q\Phi^{(0)} - \Phi^{(0)}Q - 2\dot{P}) \\ &= S_0(Q\Phi^{(0)} - \dot{P}) - \Phi^{(0)\mathrm{T}}(S_0Q - Q^{\mathrm{T}}S_0) - \Phi^{(0)\mathrm{T}}Q^{\mathrm{T}}S_0 - S_0\dot{P}. \end{aligned}$$

Subtracting from this result its transpose, we get

$$\begin{aligned} S_0\ddot{Q} - (S_0\ddot{Q})^{\mathrm{T}} = {} & 2[S_0(Q\Phi^{(0)} - \dot{P}) - (S_0(Q\Phi^{(0)} - \dot{P}))^{\mathrm{T}}] \\ & - (S_0Q - Q^{\mathrm{T}}S_0)\Phi^{(0)} - \Phi^{(0)\mathrm{T}}(S_0Q - Q^{\mathrm{T}}S_0). \end{aligned} \tag{64}$$

Similar calculations can be made in relation to (60) and (61) by repeatedly using (62), (63) and the symmetry conditions (9) and (10). The result is as follows. If we set

$$\begin{aligned} K(t) &= S_0Q(t) - Q^{\mathrm{T}}(t)S_0, \\ L(t) &= S_0(Q\Phi^{(0)} - \dot{P})(t) - (Q\Phi^{(0)} - \dot{P})^{\mathrm{T}}(t)S_0, \\ M(t) &= S_0\dot{Q}(t) + S_0P(t) + P^{\mathrm{T}}(t)S_0 - \Sigma_0, \end{aligned}$$

we find that the matrices K, L and M satisfy the differential equations

$$\begin{aligned} & \mathrm{d}^2K/\mathrm{d}t^2 = 2L - K\Phi^{(0)} - \Phi^{(0)\mathrm{T}}K, \\ & \mathrm{d}L/\mathrm{d}t = \Phi^{(0)\mathrm{T}}M^{\mathrm{T}} - M\Phi^{(0)}, \qquad \mathrm{d}M/\mathrm{d}t = L - \Phi^{(0)\mathrm{T}}K. \end{aligned}$$

In addition we know that $K(t_0) = \mathrm{d}K(t_0)/\mathrm{d}t = L(t_0) = M(t_0) = 0$, from which it follows that K, L and M must be zero for all t, which completes the proof of lemma 5.

In general, if $\mu = -(Pq + Q\dot{q})$ determines a dynamical symmetry for (42), then replacing P by $P + C$ (C constant) produces a new dynamical symmetry under the condition that

$$C\Phi^{(0)} - \Phi^{(0)}C = 0. \tag{65}$$

This is a trivial consequence of the symmetry conditions (9) and (10). With (65) in mind we now state:

Lemma 6. If $\mu=-(Pq+Q\dot{q})$ determines a *Noether symmetry* for (42) with respect to L (multiplier $S(t, q, \dot{q})$), then

$$\mu'=-[(P+C)q+Q\dot{q}]$$

with

$$C=-\tfrac{1}{2}S_0^{-1}\Sigma_0 \tag{66}$$

defines a *dynamical symmetry* for (42).

Proof. The proof consists in verifying that (65) holds true, thereby making use of the properties (62) and (63).

We are now finally in the position to prove the main theorem, concerning the reduced system (42).

Proposition 5. Let $\mu=-(Pq+Q\dot{q})$ determine a dynamical symmetry Y of (42). If Y is a Noether symmetry with respect to some Lagrangian L (determined by a multiplier $S(t, q, \dot{q})$), then there exists a related linear symmetry Y' of (42) and a related quadratic Lagrangian L_0, such that Y' is a Noether symmetry with respect to L_0.

Proof. Introducing $S_0=S(t_0, q_0, \dot{q}_0)$, we obtain a quadratic Lagrangian L_0. Defining Σ_0 as in (58), we set

$$\mu'=-[(P-\tfrac{1}{2}S_0^{-1}\Sigma_0)q+Q\dot{q}], \tag{67}$$

which according to lemma 6 yields a new *dynamical symmetry* of (42). Next, we introduce the quadratic function

$$G=\tfrac{1}{2}\dot{q}^{\mathrm{T}}S_0Q\dot{q}+q^{\mathrm{T}}(P^{\mathrm{T}}S_0-\tfrac{1}{2}\Sigma_0)\dot{q}+\tfrac{1}{2}q^{\mathrm{T}}S_0(Q\Phi^{(0)}-\dot{P})q, \tag{68}$$

where $S_0Q(t)$ and $S_0(Q\Phi^{(0)}-\dot{P})(t)$ are known to be symmetric by lemma 5. From (68) we obtain

$$\partial G/\partial\dot{q}=-S_0\mu', \tag{69}$$

$$\partial G/\partial q=(P^{\mathrm{T}}S_0-\tfrac{1}{2}\Sigma_0)\dot{q}+S_0(Q\Phi^{(0)}-\dot{P})q. \tag{70}$$

Computing Γ of both sides of (69) and taking account of (70), we obtain

$$\partial\Gamma(G)/\partial\dot{q}=(S_0\dot{Q}+S_0P+P^{\mathrm{T}}S_0-\Sigma_0)\dot{q}=0, \tag{71}$$

by (61). Equation (71) of course means that the quadratic $\Gamma(G)$ reduces to its $\dot{q}$-independent part, which explicitly reads

$$\tfrac{1}{2}q^{\mathrm{T}}[(\mathrm{d}/\mathrm{d}t)(S_0(Q\Phi^{(0)}-\dot{P}))-(P^{\mathrm{T}}S_0-\tfrac{1}{2}\Sigma_0)\Phi^{(0)}-\Phi^{(0)\mathrm{T}}(S_0P-\tfrac{1}{2}\Sigma_0)]q. \tag{72}$$

Using the differential equation (10) for P and the properties (62), (63) and (61), it is straightforward to verify that the matrix governing the quadratic expression (72) turns out to be identically zero. We therefore conclude $\Gamma(G)=0$. From (69) and the fundamental lemma 1 in the introduction it then follows that Y' is a Noether symmetry with respect to the quadratic Lagrangian L_0.

Needless to say, proposition 4 follows from proposition 5 and lemma 2.

Some final comments are in order now. The statement in proposition 4 is something one intuitively would expect to be true. Nevertheless, proving it turned out to be quite non-trivial and actually is of some interest on its own, because it brings to the surface almost all relevant aspects of our previous studies on linear systems, concerning both the search for symmetries and quadratic first integrals, and the search for multipliers in the inverse problem. This last section provides a broader platform for the results of § 3, yet it is quite independent of these results. It is worthwhile noting, however, that a complete understanding of the situation for linear systems may well be of interest for the case of arbitrary second-order systems too, because a general second-order system is derivable from a variational principle if and only if its linear variational equations have the same property.

Concerning the results stated in § 3 now, they cannot be considered as providing strong evidence for a tight connection between the inverse problem of Lagrangian mechanics and the existence of symmetries and first integrals. The least one can say is that they offer an illustration of the fact that the perfect 'Noether triangle', which was shown to exist in paper I for the case of one degree of freedom, becomes a rather problematic issue when one passes to a higher dimension. It looks likely that the existence of a dynamical symmetry and a suitably related first integral may only imply existence of a Lagrangian if one has more than just one incidence of that kind, in other words, if one studies an algebra of symmetries, as was done for instance by Takens (1977).

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Note added in proof. The proofs of lemmas 4 and 5 can be greatly simplified by noting that the matrix S_0 in (57) is identical to $S(t, q(t), \dot{q}(t))$ where $q(t)$ is the solution of (7) with the indicated initial values (see also Sarlet 1983c).

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